

Pulse shape dependence in the dynamically assisted Sauter-Schwinger effect

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While the Sauter-Schwinger effect describes nonperturbative electron-positron pair creation from vacuum by a strong and slowly varying electric field E_{strong} via tunneling, the dynamically assisted Sauter-Schwinger effect corresponds to a strong (exponential) enhancement of the pair-creation probability by an additional weak and fast electric or electromagnetic pulse E_{weak} . Using the WKB and worldline instanton method, we find that this enhancement mechanism strongly depends on the shape of the fast pulse. For the Sauter profile $1/\cosh^2(\omega t)$ considered previously, the threshold frequency ω_{crit} (where the enhancement mechanism sets in) is basically independent of the magnitude E_{weak} of the weak pulse—whereas for a Gaussian pulse $\exp(-\omega^2 t^2)$, an oscillating profile $\cos(\omega t)$ or a standing wave $\cos(\omega t) \cos(kx)$, the value of ω_{crit} does depend (logarithmically) on $E_{\text{weak}}/E_{\text{strong}}$.

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I. INTRODUCTION

In contrast to the perturbative realm of quantum field theory, where we possess well-established methods for calculating observables such as scattering cross sections, our understanding of nonperturbative effects is still rather incomplete. In quantum electrodynamics (QED), a prominent example for such a nonperturbative phenomenon is the Sauter-Schwinger effect [1–4] corresponding to the creation of electron-positron pairs out of the quantum vacuum by tunneling induced by a strong electric field E . For constant electric fields E , the leading-order pair-creation probability scales as

$$P_{e^+e^-} \sim \exp \left\{ -\pi \frac{m^2}{qE} \frac{c^3}{\hbar} \right\} = \exp \left\{ -\pi \frac{E_S}{E} \right\}, \quad (1)$$

where $\pm q$ is the charge and m the mass of the electrons and positrons. Since the critical field strength $E_S \approx 10^{18} \text{ V/m}$ is extremely large, this prediction has not been conclusively experimentally verified yet [5].

Several years ago, it was found [7] that the above pair-creation probability $P_{e^+e^-}$ can be drastically enhanced by superimposing the constant (or slowly varying) strong field E with a weaker time-dependent pulse—the dynamically assisted Sauter-Schwinger effect; see also Refs. [8–31]. Interestingly, this enhancement mechanism is already operative for frequency scales ω far below the mass gap $2mc^2$ separating the Dirac sea from the positive continuum. In the following, we study the dependence of the dynamically assisted Sauter-Schwinger effect on the shape of the additional time-dependent pulse. To this end, we employ the WKB and the worldline instanton methods in order to compare a Gaussian pulse and an oscillating or standing-wave profile with the Sauter pulse considered previously.

II. RICCATI EQUATION

Let us briefly review the main steps of the derivation (see e.g. Ref. [15] for a more detailed introduction). For simplicity, we start with the Dirac equation in 1+1 dimensions ($\hbar = c = 1$)

$$(i\gamma^\mu [\partial_\mu + iqA_\mu] + m) \cdot \psi = 0, \quad (2)$$

with the two-component spinor ψ . The purely time-dependent electric field $E(t)$ is treated as an external background field and can be described by the vector potential $A_\mu = [0, A(t)]$ via $E(t) = \dot{A}(t)$ in temporal gauge. Using the representation of the γ^μ in terms of the usual Pauli matrices $\sigma_{x,y,z}$, we get the Hamiltonian form

$$i\partial_t \psi = (-i\sigma_x \partial_x + qA(t)\sigma_x + m\sigma_z) \cdot \psi. \quad (3)$$

After a spatial Fourier transform, this gives

$$i\partial_t \psi_k = ([k + qA(t)]\sigma_x + m\sigma_z) \cdot \psi_k = \mathbf{H}_k \cdot \psi_k, \quad (4)$$

where \mathbf{H}_k is a self-adjoint matrix with the eigenvalues

$$\mathbf{H}_k \cdot \mathbf{u}_k^\pm = \pm \sqrt{m^2 + [k + qA(t)]^2} \mathbf{u}_k^\pm = \pm \Omega_k(t) \mathbf{u}_k^\pm, \quad (5)$$

and real (time-dependent) eigenvectors $\mathbf{u}_k^\pm(t)$ with the properties $(\mathbf{u}_k^\pm)^2 = 1$ and $\mathbf{u}_k^+ \cdot \mathbf{u}_k^- = 0$.

Now we expand the spinor solution $\psi_k(t)$ of Eq. (4) into these instantaneous eigenvectors

$$\psi_k(t) = \alpha_k(t) e^{-i\varphi_k(t)} \mathbf{u}_k^+(t) + \beta_k(t) e^{+i\varphi_k(t)} \mathbf{u}_k^-(t), \quad (6)$$

where we have factored out the rapidly oscillating phase

$$\varphi_k(t) = \int_{t_0}^t dt' \Omega_k(t'). \quad (7)$$

Inserting the ansatz (6) into the Dirac equation (4) and using $\dot{\mathbf{u}}_k^+ \cdot \mathbf{u}_k^+ = \dot{\mathbf{u}}_k^- \cdot \mathbf{u}_k^- = 0$, we find the evolution equations for the Bogoliubov coefficients

$$\begin{aligned} \dot{\alpha}_k(t) &= +\Xi_k(t) \beta_k(t) e^{+2i\varphi_k(t)}, \\ \dot{\beta}_k(t) &= -\Xi_k(t) \alpha_k(t) e^{-2i\varphi_k(t)}, \end{aligned} \quad (8)$$

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where we have introduced the abbreviation

$$\Xi_k(t) = \dot{\mathbf{u}}_k^+ \cdot \mathbf{u}_k^- = -\dot{\mathbf{u}}_k^- \cdot \mathbf{u}_k^+ = \frac{mq\dot{A}(t)}{2\Omega_k^2(t)}. \quad (9)$$

Note that $|\alpha_k|^2 + |\beta_k|^2$ is conserved and can be set to unity $|\alpha_k|^2 + |\beta_k|^2 = 1$ by appropriate initial conditions. Introducing the ratio $R_k(t) = \alpha_k(t)/\beta_k(t)$, we can combine the two equations (8) into one equality

$$\dot{R}_k(t) = \Xi_k(t) \left(e^{2i\varphi_k(t)} + R_k^2(t) e^{-2i\varphi_k(t)} \right), \quad (10)$$

which is the Riccati equation. In contrast to the bosonic case (where $|\alpha_k|^2 - |\beta_k|^2 = 1$), we have a plus sign within the brackets, which reflects the fermionic nature of the Dirac equation ($|\alpha_k|^2 + |\beta_k|^2 = 1$).

III. WKB ANALYSIS

In a stationary situation, the Bogoliubov coefficients α_k and β_k in Eq. (6) correspond to solutions with positive and negative energies, respectively. Thus, β_k can be interpreted as the amplitude of an electron in the Dirac sea while α_k describes an electron in the positive continuum. Assuming that the field is switched off initially and finally $E(t \rightarrow \pm\infty) = 0$, electron-positron pair creation can be understood as the transition of an electron from the Dirac sea into the positive continuum induced by the electric field $E(t)$. Thus, with the initial conditions $\alpha_k^{\text{in}} = 0$ and $\beta_k^{\text{in}} = 1$, the pair-creation probability (for the mode k) is given by

$$P_k^{e^+e^-} = |\alpha_k^{\text{out}}|^2 = \frac{|R_k^{\text{out}}|^2}{|R_k^{\text{out}}|^2 + 1}. \quad (11)$$

Note that this picture is a bit different from the usual description of particle creation in the bosonic case, where the probability is given by $|\beta_k^{\text{out}}|^2$ instead. However, for fermions, the particle-hole duality (which is the basis for the Dirac sea construction) implies that α_k and β_k behave symmetrically. Thus, one can use both representations (Dirac sea filled with either electrons or positrons) with the correct initial conditions.

Since this probability $P_k^{e^+e^-}$ is realistically very small, we may approximate $|\alpha_k^{\text{out}}| \approx |R_k^{\text{out}}| \ll 1$. For solutions with small amplitude $R_k(t) \ll 1$, the Riccati equation (10) can be approximated by

$$\dot{R}_k(t) \approx \Xi_k(t) e^{2i\varphi_k(t)}. \quad (12)$$

Note that this approximation does in general not yield the exact prefactor [33], but it does give the correct exponent—which is the quantity we are interested in.

Equation (12) can now be integrated with the initial condition $R_k^{\text{in}} = R_k(t \rightarrow -\infty) = 0$

$$R_k^{\text{out}} = R_k(t \rightarrow \infty) \approx \int_{-\infty}^{+\infty} dt \Xi_k(t) e^{2i\varphi_k(t)}. \quad (13)$$

Assuming that $A(t)$ and thus $\Xi_k(t)$ as well as $\varphi_k(t)$ are analytic functions in a strip of the complex plane including the real axis, the above integral can be calculated by deforming the integration contour into the complex t plane. In the upper complex half-plane $\text{Im}(t) > 0$, the exponential $e^{2i\varphi_k(t)}$ decays rapidly—which corresponds to the fact that R_k^{out} is exponentially suppressed. However, we cannot deform the integration contour arbitrarily far since there will be singularities, e.g., at

$$\Omega_k(t_k^*) = 0 \rightsquigarrow k + qA(t_k^*) = \pm im. \quad (14)$$

The solution of Eq. (12) can thus be estimated by

$$R_k^{\text{out}} \sim \exp\{2i\varphi_k(t_k^*)\} \sim \exp\{-2\text{Im}[\varphi_k(t_k^*)]\}. \quad (15)$$

Let us consider the most simple example: a constant electric field $E = \text{const}$. In this case, we have $A(t) = Et$ and thus we only get two singularities

$$t_k^* = \frac{\pm im - k}{qE}. \quad (16)$$

Insertion into the estimate (15) then yields the usual Schwinger exponent in Eq. (1).

In the more general case with more singularities t_k^* , they all contribute to the estimate (15) but typically the one with the smallest $|\text{Im}[\varphi_k(t_k^*)]|$ dominates (for a more detailed discussion, cf. Appendix B). It can happen, however, that two (or more) have comparable magnitude—in which case there can be interference effects, which have already been observed in Refs. [9, 15–17, 34, 35], for example.

IV. SAUTER PROFILE

Let us first apply the method sketched above to the case of a Sauter pulse [1] considered previously in Refs. [7, 17]. More specifically, we consider an electric field of the form

$$E(t) = E_1 + \frac{E_2}{\cosh^2(\omega_2 t)}, \quad (17)$$

where $E_2 \ll E_1 \ll E_S$ and $\omega_2 \ll m$. The associated vector potential reads

$$A(t) = E_1 t + \frac{E_2}{\omega_2} \tanh(\omega_2 t), \quad (18)$$

and has poles at $\omega_2 t = i\pi/2 + i\pi\mathbb{Z}$. In terms of the dimensionless quantities $\tau = \omega_2 t$, $\varepsilon = E_2/E_1$, $p = k/m$, as well as the combined Keldysh parameter [7, 36]

$$\gamma = \frac{m\omega_2}{qE_1}, \quad (19)$$

the equation (14) for the singularities reads

$$\tau^* + \varepsilon \tanh \tau^* = \gamma(\pm i - p). \quad (20)$$

Consequently, for small ε , we basically get the same singularity (16) in the upper half-plane as for a constant field, which we call the regular singularity

$$\tau_{\text{reg}}^* = \gamma(i - p) + \mathcal{O}(\varepsilon). \quad (21)$$

However, from the analytic continuation of the vector potential (18), we get additional singularities from the poles of the complex tanh function, for example at

$$\tau_{\text{add}}^* = \frac{\pi}{2} i. \quad (22)$$

Now, setting $p = 0$ for simplicity, we have the following picture: For small Keldysh parameters [Eq. (19)], the regular singularity is closer to the real axis and thus its contribution (15) dominates. In this case, we obtain basically the same exponent as for the constant strong field E_1 alone (ordinary Sauter-Schwinger effect). If the Keldysh parameter exceeds the critical (threshold) value of

$$\gamma_{\text{crit}} = \frac{\pi}{2}, \quad (23)$$

however, the additional singularity $\tau_{\text{add}}^* = i\pi/2$ caused by the Sauter pulse $E_2/\cosh^2(\omega_2 t)$ is closer to the real axis and thus its contribution starts to dominate. Thus, the exponent in Eq. (15) is reduced and hence the probability enhanced (dynamically assisted Sauter-Schwinger effect). Inserting the singularity τ^* into the estimate (15) yields

$$\text{Im}[\varphi_k(t_k^*)] \approx \frac{E_S}{2E_1} \text{Im}\left[\phi\left(\frac{\tau^*}{\gamma} + p\right)\right], \quad (24)$$

with the auxiliary function [7, 17]

$$\phi(z) = z\sqrt{1+z^2} + \text{arsinh } z. \quad (25)$$

For $\gamma < \gamma_{\text{crit}}$, the regular singularity $\tau_{\text{reg}}^* \approx \gamma(i - p)$ dominates and we get $\phi(i) = i\pi/2$ which reproduces Eq. (1). For $\gamma > \gamma_{\text{crit}}$, on the other hand, we have to insert $\tau_{\text{add}}^* = i\pi/2$ which yields the results in Ref. [7]. For Keldysh parameters near the critical value (23) the competition between regular and additional singularities leads to p -dependent interference effects [16, 17].

V. GAUSS PULSE

Now let us repeat the same analysis for another bell-shaped curve which is visually almost indistinguishable from the Sauter profile $1/\cosh^2(\omega_2 t)$: a Gauss pulse

$$E(t) = E_1 + E_2 \exp\{-(\omega_2 t)^2\}, \quad (26)$$

with the same hierarchy as before, i.e., $E_2 \ll E_1 \ll E_S$ and $\omega_2 \ll m$ (see Ref. [19] for some numeric results). The vector potential can be expressed in terms of the error function

$$A(t) = E_1 t + \frac{\sqrt{\pi} E_2}{2\omega_2} \text{erf}(\omega_2 t). \quad (27)$$

As an important difference to the Sauter profile, the Gauss pulse and the error function are analytic in the entire complex plane and thus do not give rise to additional singularities. Hence the only singularities are given by Eq. (14)

$$\tau^* + \varepsilon \frac{\sqrt{\pi}}{2} \text{erf } \tau^* = \gamma(\pm i - p). \quad (28)$$

Again, for small and moderate values of γ , we basically obtain the same regular singularity as in Eqs. (16) and (21) and thus the same exponent as for the constant strong field E_1 alone (ordinary Sauter-Schwinger effect). Moreover, there appears an infinite set of additional singularities further away from the real axis whose contributions are therefore suppressed.

If γ becomes large enough, however, the regular singularity starts to deviate from its ordinary position (21) while its imaginary part approaches that of the additional singularities (for more details, cf. Appendix A 1). Setting $p = 0$ for simplicity, the asymptotic behavior of the error function for large imaginary arguments [37],

$$\text{erf}(i\gamma) \sim \frac{\exp\{\gamma^2\}}{\gamma\sqrt{\pi}}, \quad (29)$$

implies that this motion begins when $\exp\{\gamma^2\}$ is large enough to compensate the smallness of ε , i.e., the critical value of the Keldysh parameter roughly scales as

$$\gamma_{\text{crit}} \sim \sqrt{|\ln \varepsilon|}. \quad (30)$$

As a result, although the Gauss pulse looks very similar to the Sauter pulse, they behave remarkably differently regarding the dynamically assisted Sauter-Schwinger effect. In one case (Sauter pulse), the critical frequency where the enhancement sets in is basically independent of $\varepsilon = E_2/E_1$ whereas in the other case (Gaussian), this threshold frequency scales as $\omega_2^{\text{crit}} \sim \sqrt{|\ln \varepsilon|}$.

Apart from the motion of the singularities, the additional Gaussian pulse does also affect the phase function φ_k in Eqs. (7) and (15). Again setting $p = 0$ for simplicity, we get to first order in ε

$$\text{Im}[\varphi_k(t_k^*)] \approx \frac{\pi E_S}{4E_1} \left[1 - \varepsilon e^{\gamma^2/2} \left(I_0\left[\frac{\gamma^2}{2}\right] - I_1\left[\frac{\gamma^2}{2}\right] \right) \right], \quad (31)$$

where I_ν are the modified Bessel functions of the first kind. Note, however, that this expansion is only reliable for Keldysh parameters well below the critical value (30).

VI. OSCILLATION

As our third example, let us consider a harmonically oscillating weak field (see also Refs. [12, 18, 19, 22, 27, 29])

$$E(t) = E_1 + E_2 \cos(\omega_2 t), \quad (32)$$

again with $E_2 \ll E_1 \ll E_S$ and $\omega_2 \ll m$. Similar to the Gaussian case, the vector potential

$$A(t) = E_1 t + \frac{E_2}{\omega_2} \sin(\omega_2 t), \quad (33)$$

is analytic in the entire complex plane. Again Eq. (14) yields the singularities

$$\tau^* + \varepsilon \sin \tau^* = \gamma(\pm i - p). \quad (34)$$

The solutions of this transcendental equation can be determined graphically (cf. Appendix A 2). If we again set $p = 0$ for simplicity, the dominant, regular singularity is purely imaginary and given by

$$\text{Im } \tau^* + \varepsilon \sinh(\text{Im } \tau^*) = \gamma. \quad (35)$$

As in the Gaussian case, small and moderate values of γ basically do not affect the singularity (since $\varepsilon \ll 1$) and the dynamically assisted Sauter-Schwinger effect requires γ to be large enough such that the magnitude of $\sinh \gamma$ compensates the smallness of ε . In this limit, we may approximate $\sinh \gamma \approx e^\gamma/2$ and thus the critical value of the Keldysh parameter roughly scales as

$$\gamma_{\text{crit}} \sim |\ln \varepsilon|. \quad (36)$$

Again, we may Taylor expand the phase function φ_k in Eqs. (7) and (15) for Keldysh parameters well below this critical value (36)

$$\text{Im} [\varphi_k(t_k^*)] \approx \frac{\pi E_S}{4E_1} \left[1 - 2\varepsilon \cos(p\gamma) \frac{I_1(\gamma)}{\gamma} \right]. \quad (37)$$

VII. WORLDLINE INSTANTON METHOD

It is illuminating to address the above scenarios with another technique, the worldline instanton method [34, 38–45]. In contrast to the WKB approach, the worldline instanton technique does not yield the momentum dependence (it only provides the total pair-creation probability), but it has the advantage that some calculations are easier to do. Again, let us begin with a brief review of this method. The total pair-creation probability $P_{e^+e^-}$ is related to the vacuum persistence amplitude $\langle 0_{\text{out}} | 0_{\text{in}} \rangle$

$$P_{e^+e^-} = 1 - |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2. \quad (38)$$

If we now express the vacuum persistence amplitude $\langle 0_{\text{out}} | 0_{\text{in}} \rangle$ in terms of the path integral and perform a series of transformations and approximations, we find that $P_{e^+e^-}$ can be estimated in analogy to Eq. (15) via

$$P_{e^+e^-} \sim e^{-\mathcal{A}}. \quad (39)$$

Here \mathcal{A} is the worldline instanton action in the presence of the electric field E represented by the Euclidean vector potential $A_\mu(x^\nu)$ after analytic continuation to imaginary time $x_\mu = (x, y, z, it) = (x_1, x_2, x_3, x_4)$

$$\mathcal{A} = ma + iq \int_0^1 du \dot{x}^\mu A_\mu(x^\nu[u]). \quad (40)$$

The worldline instanton $x_\mu(u)$ is a closed loop in Euclidean space-time satisfying the boundary conditions $x_\mu(0) = x_\mu(1)$ and the instanton equations of motion

$$m\ddot{x}_\mu = iq a F_{\mu\nu} \dot{x}_\nu, \quad (41)$$

with $\dot{x}_\mu = dx_\mu/du$ and $\ddot{x}_\mu = d^2x_\mu/du^2$. Since $F_{\mu\nu}$ is antisymmetric, the absolute value of the “four-velocity”

$$\dot{x}_\mu \dot{x}^\mu = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 = a^2, \quad (42)$$

is conserved.

A. Integral form of the instanton action

In certain cases, for example a purely time-dependent electric field $\mathbf{E} = E(t)\mathbf{e}_z$, the instanton action \mathcal{A} can be computed in an especially simple way. Let us consider a Euclidean four-potential of the form

$$iA_3(x_4) = \frac{E}{\omega} f(\omega x_4), \quad (43)$$

where $f(z)$ is a dimensionless odd analytic function such that the electric field $\mathbf{E} = f'(\omega t)\mathbf{e}_z$ is real. Inserting the vector potential (43), the only nonvanishing components of $F_{\mu\nu}$ are $iF_{43} = -iF_{34} = E f'(\omega x_4)$ and the instanton equations (41) reduce to

$$\ddot{x}_3 = -\frac{qEa}{m} f'(\omega x_4) \dot{x}_4, \quad (44a)$$

$$\ddot{x}_4 = +\frac{qEa}{m} f'(\omega x_4) \dot{x}_3, \quad (44b)$$

as well as $\ddot{x}_1 = \ddot{x}_2 = 0$. Thus, in order to have closed loops $x_\mu(0) = x_\mu(1)$, we must have $\dot{x}_1 = \dot{x}_2 = 0$ and hence Eq. (42) simplifies to

$$a^2 = \dot{x}_3^2 + \dot{x}_4^2. \quad (44c)$$

Equation (44a) can be integrated immediately to give

$$\dot{x}_3 = -\frac{a}{\gamma} f(\omega x_4), \quad (45)$$

where we have set the integration constant to zero in order to get closed loops. Using (44c), we arrive at

$$\dot{x}_4 = \pm \sqrt{a^2 - \dot{x}_3^2} = \pm a \sqrt{1 - \frac{1}{\gamma^2} f^2(\omega x_4)}. \quad (46)$$

Furthermore, for a field of the form (43), the instanton action (40) can be simplified as well

$$\begin{aligned} \mathcal{A} &\stackrel{(43)}{=} ma + \frac{qE}{\omega} \int_0^1 du f(\omega x_4) \dot{x}_3 \\ &\stackrel{(45)}{=} ma - \frac{m}{a} \int_0^1 du \dot{x}_3^2 \stackrel{(44c)}{=} \frac{m}{a} \int_0^1 du \dot{x}_4^2. \end{aligned} \quad (47)$$

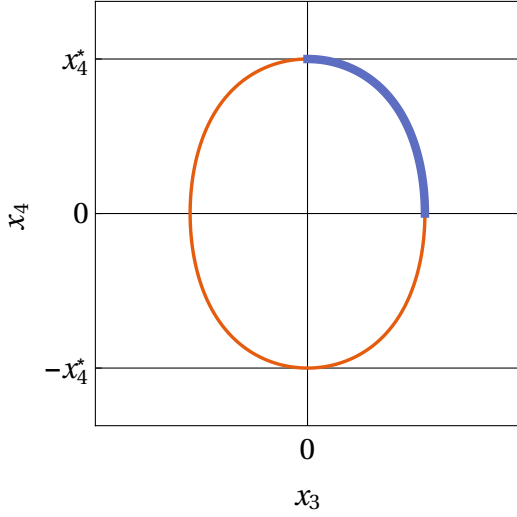


FIG. 1: Example of an instanton trajectory $x_\mu(u)$ with the appropriate symmetries. In this case, the full instanton action \mathcal{A} is four times the action along the highlighted (or any) quarter of the trajectory.

Since $f(x_4\omega)$ is an odd function and thus $f'(x_4\omega)$ and $f^2(x_4\omega)$ are even functions, the instanton trajectory has a fourfold symmetry: $x_3 \rightarrow -x_3$ and $x_4 \rightarrow -x_4$. As a result, the contribution to the action (47) is the same for each quarter of the trajectory; see Fig. 1. In each of these quarters, \dot{x}_4 is a unique function [Eq. (46)] of x_4 and thus we can turn the u integral (47) into an x_4 integral

$$\begin{aligned} \mathcal{A} &= 4 \frac{m}{a} \int_0^{1/4} du \dot{x}_4^2 \stackrel{(46)}{=} 4m \int_{x_4(0)}^{x_4(1/4)} dx_4 \sqrt{1 - \frac{1}{\gamma^2} f^2(\omega x_4)} \\ &= 4 \frac{m^2}{qE} \int_0^{\chi^*} d\chi \sqrt{1 - \frac{1}{\gamma^2} f(\gamma\chi)^2}, \end{aligned} \quad (48)$$

where we have substituted $\chi = qEx_4/m$ in the last line. The integral coincides (up to substitutions) with the function $g(\gamma^2)$ in Ref. [42]. The turning point χ^* is given implicitly by the condition

$$\gamma = f(\gamma\chi^*), \quad (49)$$

which corresponds to t_k^* in Eq. (14) for $p = 0$, where we have $\tau^* = i\gamma\chi^*$. The expression (48) is generally much easier to evaluate numerically—and approximate analytically—than solving the equations of motion (41).

VIII. DYNAMICAL ASSISTANCE

In order to study the dynamically assisted Sauter-Schwinger effect described above, we choose the dimensionless function $f(\chi)$ from Sec. VII A to be

$$f(\chi) = \chi + \varepsilon g(\chi), \quad (50)$$

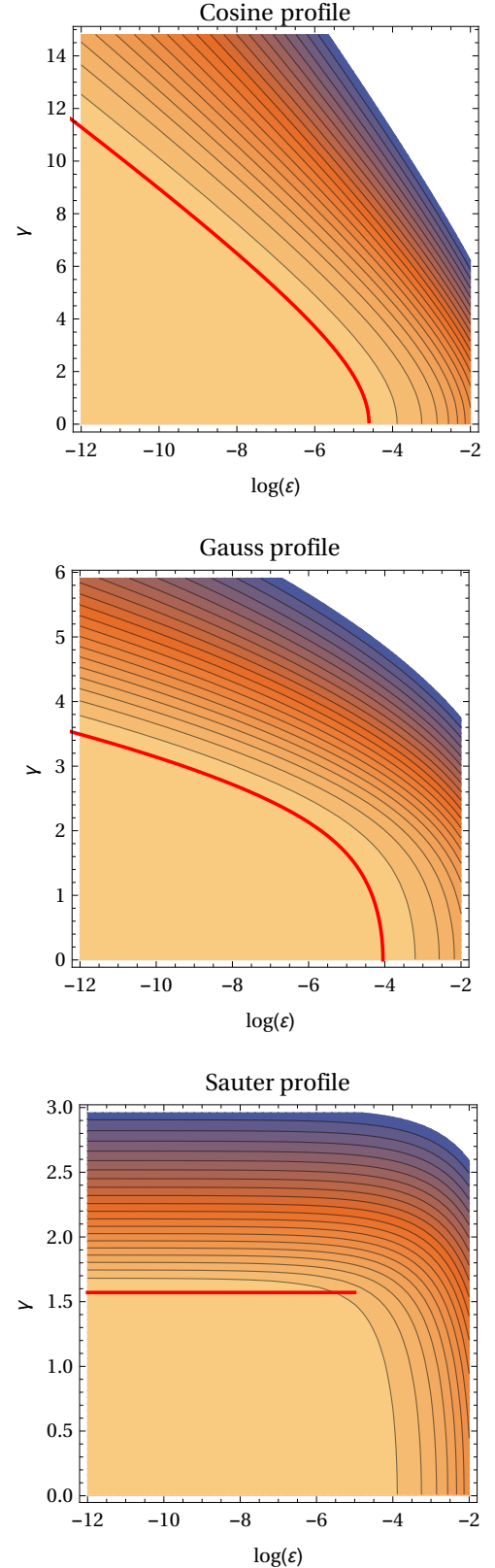


FIG. 2: Instanton action \mathcal{A} ranging from $\pi E_S/E$ (light yellow) to $2E_S/E$ (dark blue) for the considered profiles as a function of γ and ε . Thin (black) lines are contour lines and the additional thick (red) lines depict the corresponding estimates (57), (54), and (23), respectively. Note the different scales of the γ axes.

with $\varepsilon \ll 1$ and a function $g(\chi)$ representing the additional weak time-dependent profile

$$\mathbf{E} = E [1 + \varepsilon g'(i\omega t)] \mathbf{e}_z. \quad (51)$$

Assuming that $g(\chi)$ is a well-behaved function (which is the case for the Gaussian and the oscillating profile—but not for the Sauter pulse due to its singularities), we can expand the integral (48) to first order in ε

$$\begin{aligned} \mathcal{A} &= 4 \frac{m^2}{qE} \int_0^{x^*} d\chi \sqrt{1 - \left(\chi + \varepsilon \frac{g(\gamma\chi)}{\gamma} \right)^2} \\ &\approx 4 \frac{m^2}{qE} \left(\int_0^1 d\chi \sqrt{1 - \chi^2} - \varepsilon \int_0^1 \frac{d\chi \chi}{\sqrt{1 - \chi^2}} \frac{g(\gamma\chi)}{\gamma} \right) \\ &= \frac{m^2\pi}{qE} \left(1 - \frac{4\varepsilon}{\pi\gamma} \int_0^1 d\xi g(\gamma\sqrt{1 - \xi^2}) \right) \\ &= \pi \frac{E_S}{E} \left(1 - \frac{4\varepsilon}{\pi\gamma} G(\gamma) \right). \end{aligned} \quad (52)$$

To first order in ε , the dependence of χ^* on ε does not contribute because the integrand vanishes at that point. The contribution to zeroth order in ε reproduces the constant field in Eq. (1). The form of $G(\gamma)$ then determines how much, for an additional pulse $\varepsilon g'(i\omega t)$, a given strength ε and Keldysh parameter γ , the instanton action \mathcal{A} is reduced and thus pair production improved.

A. Gaussian pulse

For a Gaussian pulse, the Euclidean vector potential is represented by the imaginary error function

$$g(\chi) = \frac{\sqrt{\pi}}{2} \operatorname{erfi} \chi, \quad (53)$$

and thus the function $G(\gamma)$ in Eq. (52) reads

$$G(\gamma) = \frac{\pi\gamma}{4} e^{\gamma^2/2} \left(I_0 \left[\frac{\gamma^2}{2} \right] - I_1 \left[\frac{\gamma^2}{2} \right] \right). \quad (54)$$

Comparison with Eq. (31) shows agreement for $p = 0$, which demonstrates that the pairs are produced predominantly with $p = 0$ (as expected). Of course, this is no accidental coincidence as the manipulations in Eq. (52) are equivalent to those leading to Eqs. (31) and (37) for $p = 0$; see also Ref. [26].

For large γ , the function $G(\chi)$ behaves as $\exp\{\gamma^2\}/\gamma^2$ and thus the threshold for the dynamically assisted Sauter-Schwinger effect is reached when

$$\varepsilon \sim \gamma^3 e^{-\gamma^2}, \quad (55)$$

which is consistent with Eq. (30). Again, we would like to stress that the first-order expansion (54) is only valid for Keldysh parameters γ well below this critical value.

B. Oscillating profile

In order to represent an oscillation, we choose

$$g(\chi) = \sinh \chi. \quad (56)$$

In this case, the $G(\gamma)$ in Eq. (52) becomes

$$G(\gamma) = \frac{\pi}{2} I_1(\gamma). \quad (57)$$

Again, comparison with Eq. (37) shows agreement for $p = 0$ as well as for $p = \pm 2\pi/\gamma$ and $p = \pm 4\pi/\gamma$ etc. These additional p values result from the periodicity of the oscillation which implies that the times t and $t \pm T$ with $T = 2\pi/\omega$ are equivalent. During one period T , the acceleration by the strong electric field E causes a momentum shift of $\Delta k = qET = qE2\pi/\omega = 2\pi m/\gamma$, i.e., $\Delta p = 2\pi/\gamma$.

In view of the asymptotic behavior $G(\gamma) \sim \gamma^{3/2} e^{-\gamma}$ for large γ , the threshold for the dynamically assisted Sauter-Schwinger effect is reached when

$$\varepsilon \sim \gamma^{3/2} e^{-\gamma}, \quad (58)$$

which is again consistent with Eq. (36).

IX. STANDING WAVE

One might perhaps object that the fields $\mathbf{E}(t)$ considered here are not solutions of the Maxwell equations in vacuum, i.e., without sources. This objection could be motivated by the idea that the fermionic modes somehow “feel” the sources generating the electromagnetic background field and thus behave totally different in field configurations with and without sources. However, retracing the steps of the formalism presented above, we see that we only used the Dirac equation in a given external background field A_μ . Whether this field A_μ is a solution of the Maxwell or Proca or Yang-Mills equations with or without sources is irrelevant for our derivation. Note that we also did not exploit local gauge invariance.

In order to illuminate this point, let us consider the following electric field profile:

$$\mathbf{E}(t, x) = [E_1 + E_2 \cos(\omega t) \cos(kx)] \mathbf{e}_z. \quad (59)$$

It can be represented by the Euclidean vector potential

$$iA_3(x_1, x_4) = E_1 x_4 + \frac{E_2}{\omega} \sinh(\omega x_4) \cos(kx_1), \quad (60)$$

which generates the above electric field in z direction as well as a magnetic field in the y direction. Thus, the only nonvanishing components of $F_{\mu\nu}$ are

$$iF_{43} = -iF_{34} = E_1 + E_2 \cosh(\omega x_4) \cos(kx_1), \quad (61a)$$

$$iF_{13} = -iF_{31} = -E_2 \frac{k}{\omega} \sinh(\omega x_4) \sin(kx_1), \quad (61b)$$

which yields the following instanton equations:

$$m\ddot{x}_1 = -qaE_2 \frac{k}{\omega} \sinh(\omega x_4) \sin(kx_1) \dot{x}_3, \quad (62a)$$

$$m\ddot{x}_3 = -qa[E_1 + E_2 \cosh(\omega x_4) \cos(kx_3)] \dot{x}_4 + qaE_2 \frac{k}{\omega} \sinh(\omega x_4) \sin(kx_1) \dot{x}_1, \quad (62b)$$

$$m\ddot{x}_4 = qa[E_1 + E_2 \cosh(\omega x_4) \cos(kx_3)] \dot{x}_3. \quad (62c)$$

The equation (62a) for the transverse component x_1 is solved by $x_1(u) = 0$ or $x_1(u) = \pm\pi/k$ etc. Choosing a solution in the maximum of the electric field such as $x_1(u) = 0$, the other equations reduce to

$$m\ddot{x}_3 = -qa[E_1 + E_2 \cosh(\omega x_4)] \dot{x}_4, \quad (63a)$$

$$m\ddot{x}_4 = qa[E_1 + E_2 \cosh(\omega x_4)] \dot{x}_3. \quad (63b)$$

As a result, we find that the instanton trajectories and thus also the instanton action \mathcal{A} are not affected by the additional factor $\cos(kx)$ in Eq. (59) at all. This invariance can be generalized to vector potentials $A_3(x_1, x_4)$ for which $\partial A_3/\partial x_1$ vanishes at $x_1 = 0$ for all x_4 , which is the case for all even functions $A_3(-x_1, x_4) = A_3(x_1, x_4)$, for example.

Previous experience suggests that temporal variations of the electric field $E(t)$ tend to increase the pair-creation probability whereas longitudinal spatial variations $E(x)e_x$ tend to decrease it (an interplay between these two tendencies is discussed in Ref. [30]; see also Ref. [46]). Our example shows that a transversal spatial dependence does not necessarily have this effect because the worldline instanton action \mathcal{A} is actually independent of k . Note, however, that the situation would change drastically for a propagating wave such as $\cos(\omega t - kx)$ instead of a standing wave $\cos(\omega t) \cos(kx)$; see, e.g., Ref. [10]. Furthermore, the prefactor in front of the exponential in Eq. (39), which can be derived via studying small perturbations around the instanton trajectory, could well depend on k since it affects the effective pair-creation volume.

For $\omega = k$, the field (59) represents a vacuum solution of the Maxwell equations in the form of a standing wave, which can be generated by the superposition of two plane waves $\cos(\omega t \pm kx)$ with the same polarization but propagating into opposite directions. For $\omega \neq k$, however, this field profile only solves the Maxwell equations with nonzero sources. Nevertheless, the instanton action \mathcal{A} is the same for $\omega = k$ and $\omega \neq k$, which demonstrates that it is irrelevant whether the background field A_μ is a vacuum solution of the Maxwell equations or not. The difference between a standing $\cos(\omega t) \cos(kx)$ and a propagating wave $\cos(\omega t - kx)$ is far more important than the question of whether the field is a vacuum solution of the Maxwell equations ($\omega = k$) or not ($\omega \neq k$).

Of course, we do not deny that it is certainly desirable to calculate the pair-creation probability for field configurations which are as realistic as possible. On the other hand, our results show that one should not discard results

just because the considered field profile is not a vacuum solution of the Maxwell equations [47].

X. CONCLUSIONS

Via the WKB and the worldline instanton method, we studied and compared the dynamically assisted Sauter-Schwinger effect for three profiles (Sauter, Gauss, and oscillating) and found qualitative differences. For the Sauter pulse considered previously, the critical (threshold) Keldysh parameter (23) is determined by the competition between the regular singularity (21) and the additional singularity (22) and is basically independent of the magnitude E_2 of the weak field. By contrast, the other two cases do not feature such a competition between different singularities and the critical (threshold) Keldysh parameters (30) and (36) do depend on the magnitude of the weak field, albeit only logarithmically. These profound differences can be traced back to the distinct features of the three profiles in the complex t plane—which also determines the momentum dependence including interference effects etc.

Furthermore, one would expect that these differences do also affect the behavior of the prefactor (which was not considered here) in front of the exponential in Eqs. (15) and (39), e.g., its dependence on ε . Note, however, that this prefactor cannot resolve or counterbalance the observed difference between the pulse profiles (e.g., Sauter and Gauss). For example, let us consider a Keldysh parameter a bit above the threshold value (23) of $\gamma_{\text{crit}} = \pi/2$. For a Sauter pulse, the dynamically assisted Sauter-Schwinger effect is already operational in this regime—which has also been confirmed by numerical computations of the total pair-creation amplitude (including the prefactor); see, e.g., Ref. [16]. For a Gaussian profile with the same parameters, on the other hand, the weak-field contribution $\propto E_2$ is negligible in the vicinity of the instanton trajectory for small values of ε . Since the prefactor can be determined by considering small perturbations around the instanton trajectory, it is thus also almost unaffected by this weak-field contribution $\propto E_2$. This behavior was confirmed by preliminary numerical investigations [49] which indicate that the Gaussian prefactor is approximately constant (i.e., independent of ε and γ) for small $\gamma \leq 1$ but then starts to grow with both, increasing ε and γ , for larger values of γ . By contrast, the prefactor of the Sauter pulse is nearly independent of ε for small as well as large γ and only varies with ε near threshold $\gamma \approx \gamma_{\text{crit}} = \pi/2$. (Note that one cannot simply take the limit $\varepsilon \rightarrow 0$ here since the instanton approximation would break down eventually.) In both cases, however, the moderate variation of the prefactor is subdominant in comparison to the strong enhancement of the pair-creation probability induced by the changes in the exponent considered here.

Our results can be generalized in a straightforward manner to other profiles, for example a Lorentzian pro-

file, which is expected to behave similarly to the Sauter pulse. The fact that two visually almost indistinguishable bell-shaped pulses (Sauter and Gaussian) display such drastic differences demonstrates that tunneling in time-dependent backgrounds can show surprising and partly counterintuitive results—which motivates further studies in order to understand these nonperturbative phenomena better.

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Appendix A: Positions of the singularities t_k^*

The outgoing ratio R_k^{out} , which determines the pair-creation probability, is approximated by the contour integral in Eq. (13) according to the WKB method. In order to find a suitable integration contour in the upper complex half-plane $\text{Im}(t) > 0$, we need to know the positions of the integrand's singularities t_k^* given by Eq. (14). These singularities are poles of the function $\Xi_k(t)$ as well as square-root-type branch points (with associated cuts) of $\varphi_k(t)$. The weak electric field may introduce further complex singularities into the vector potential $A(t)$ itself and thus into the integrand; however, this is neither the case for the Gauss pulse nor for the oscillation.

1. Gauss pulse

The dimensionless singularity positions $\tau^* = \omega_2 t_k^*$ are given by Eq. (28) in this case. The regular solution τ_{reg}^* [cf. Eq. (21)], which is the only solution to stay finite in the limit $\varepsilon \rightarrow 0$, can be Taylor expanded around $\varepsilon = 0$

$$\tau_{\text{reg}}^* = \gamma(i - p) - \frac{\sqrt{\pi}}{2} \text{erf}[\gamma(i - p)]\varepsilon + \frac{\sqrt{\pi}}{2} e^{-\gamma^2(i-p)^2} \times \text{erf}[\gamma(i - p)]\varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (\text{A1})$$

For values of γ below the threshold $\gamma_{\text{crit}} \approx \sqrt{|\ln \varepsilon|}$, the terms of order $\mathcal{O}(\varepsilon)$ and higher are much smaller than unity. This is no longer true, however, as γ approaches γ_{crit} . Hence, the truncated Taylor series (A1) is appropriate to describe the movement of the regular singularity as the parameters are varied within the subcritical regime.

The other solutions of the singularity equation (28) diverge as ε approaches zero, so we may approximate the error function by its asymptotic form $\text{erf} \tau \sim -\exp(\tau^2)/(\sqrt{\pi}\tau)$ [37] for small ε in order to describe

these additional singularities. In this limit, the τ^* -independent terms on the right-hand side of Eq. (28) can be neglected, and we find the series of singularities

$$\tau_{\text{add},n}^* \approx \sqrt{\frac{\ln \varepsilon}{\cos(2\theta_n)}} e^{i\theta_n} \quad (\text{A2})$$

with $n \in \mathbb{Z}$. The complex phase θ_n is the single real solution of the transcendental equation

$$|\ln \varepsilon| \tan(2\theta_n) - 2\theta_n = 2\pi n \quad (\text{A3})$$

in the range $\pi/4 < \theta_n < 3\pi/4$, which can be found numerically. Note that, in this approximation, the dependence on momentum p and the difference between the plus and the minus cases in Eq. (28) disappear. We thus only get one result where there are actually two different singularities (\pm cases). For fixed ε , all points $\tau_{\text{add},n}^*$ lie on a hyperbola in the complex plane since $\text{Im}^2(\tau_{\text{add},n}^*) - \text{Re}^2(\tau_{\text{add},n}^*) = |\ln \varepsilon|$ for all n . If we consider a fixed n instead, we can approximate the solution of Eq. (A3) for sufficiently large $|\ln \varepsilon|$ by linearizing the tangent around $2\theta_n = \pi$. To leading order, this gives the asymptotic form

$$\tau_{\text{add},n}^* \approx -\frac{\pi(n + 1/2)}{\sqrt{|\ln \varepsilon|}} + i\sqrt{|\ln \varepsilon|}, \quad (\text{A4})$$

which parametrizes another hyperbola as ε is varied, cf. Fig. 3(b).

For $p = 0$, the singularity equation (28) is symmetric with respect to the imaginary axis ($\text{Re} \tau^* \rightarrow -\text{Re} \tau^*$) and the regular singularity remains on this axis for all values of ε and γ . The movement of the singularities is depicted in Fig. 3.

2. Oscillation

a. Graphical solution of the singularity equation

The transcendental equation (34) for the dimensionless singularities τ^* can be solved graphically. We start to develop the graphical solution scheme by splitting the equation and the complex variable $\tau^* = u^* + iv^*$ into real and imaginary parts

$$u^* + \varepsilon \sin(u^*) \cosh(v^*) = -p\gamma, \quad (\text{A5})$$

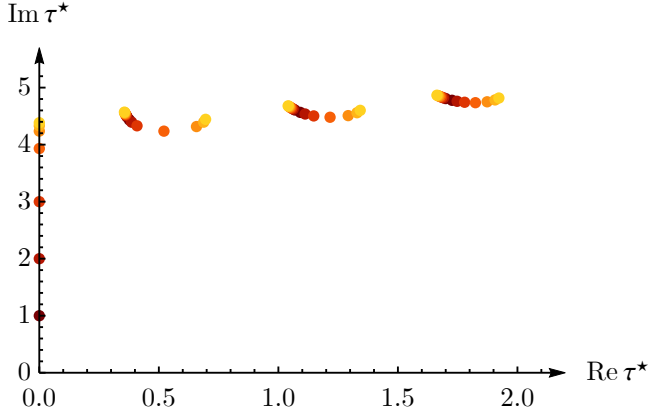
$$v^* + \varepsilon \cos(u^*) \sinh(v^*) = \pm\gamma. \quad (\text{A6})$$

For singularities in the upper complex half-plane ($v^* > 0$) with $\sin u^* \neq 0$, Eq. (A5) can be solved uniquely for the imaginary part v^* , which yields $v^* = v_{\text{sing}}(u^*)$ with

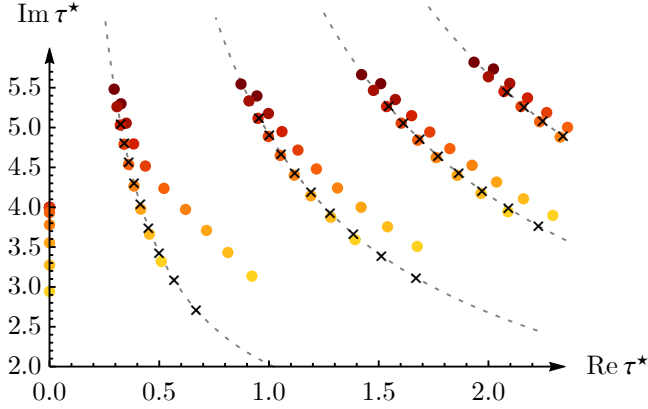
$$v_{\text{sing}}(u) = \text{arcosh}\left(-\frac{u + p\gamma}{\varepsilon \sin u}\right). \quad (\text{A7})$$

Inserting this expression into Eq. (A6) leads to

$$F(u^*) = \pm 1 \quad (\text{A8})$$



(a) Singularities moving as γ is varied from 1 to 7 (dark red to light yellow dots) for constant $\varepsilon = 10^{-7}$. The associated threshold for the dynamic assistance is $\sqrt{|\ln \varepsilon|} \approx 4$. In the subcritical γ range, the regular singularity on the imaginary axis moves essentially as in the case of the ordinary Sauter-Schwinger effect ($\tau_{\text{reg}}^* \approx i\gamma$), cf. Eq. (16), and the additional singularities are much deeper in the complex plane. As γ gets close to the critical threshold, the regular singularity slows down, cf. Eq. (A1), and “pushes” the additional singularities away from the real axis.

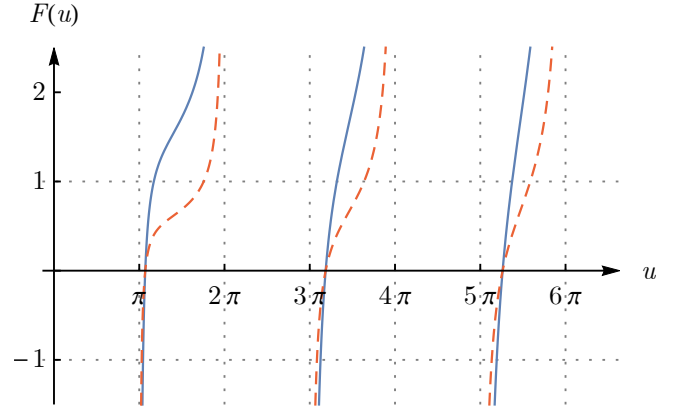


(b) Singularities for $\gamma = 4$ and $\varepsilon = 10^s$ with s going from -11 to -3 (dark red to light yellow dots). The black crosses are the corresponding asymptotic solutions $\tau_{\text{add},n}^*$ for $n = -1, \dots, -4$, see Eq. (A2), which move along the dashed hyperbolas as ε is varied. The transition to the critical regime is at $\varepsilon \approx 10^{-7}$. At this value, the regular singularity on the imaginary axis starts to move towards the real axis. The additional singularities approach the real axis continuously as ε grows. In the subcritical range, the asymptotic values $\tau_{\text{add},n}^*$ are appropriate to describe the movement of the additional singularities.

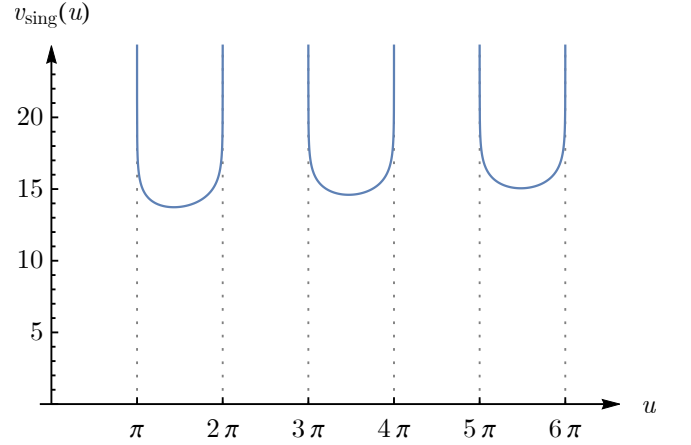
FIG. 3: Numerical solutions τ^* of the Gauss-case singularity equation (28) for the momentum $p = 0$.

with the function

$$F(u) = \frac{1}{\gamma} \left[\text{arcosh} \left(-\frac{u + p\gamma}{\varepsilon \sin u} \right) + \cos(u) \sqrt{\left(\frac{u + p\gamma}{\sin u} \right)^2 - \varepsilon^2} \right]. \quad (\text{A9})$$



(a) Graphical solution of Eq. (A8). The function $F(u)$ is plotted for $\gamma = 8$ (solid, subcritical) and $\gamma = 20$ (dashed, supercritical).



(b) Imaginary parts of the singularities over their real parts.

FIG. 4: Graphical solution of the singularity equation (34) for $p = 0$ and $\varepsilon = 10^{-5}$, so γ_{crit} is roughly $|\ln \varepsilon| \approx 11.5$, cf. Eq. (36). The function $F(u)$ is even for $p = 0$, so we concentrate on positive u for simplicity. The real parts u^* of the singularities τ^* are determined in the upper plot (a). For $p = 0$, $F(u > 0)$ is only real for $u \in (\pi, 2\pi), (3\pi, 4\pi)$, etc. and increases strictly over each one of these intervals, respectively, so there are two singularities per interval. The corresponding imaginary parts v^* are found by evaluating $v_{\text{sing}}(u)$ shown in the lower plot (b) at u^* .

The real equation (A8) can be solved graphically (see Fig. 4) in order to find all singularities τ^* with real parts satisfying $\sin u^* \neq 0$. The corresponding imaginary parts of the singularities are given by $v_{\text{sing}}(u^*)$.

Let us now treat the special case where $\sin u^* = 0$, i.e., those singularities with real parts $u^* \in \pi\mathbb{Z}$, which are not covered by the graphical solution scheme. Then, Eq. (A5) reduces to $u^* = -p\gamma$, so this case can only occur for particular momenta $p \in \mathbb{Z}\pi/\gamma$. The cosine in the lower Eq. (A6) is $\cos u^* = \pm 1$ for these singularities. For $\cos u^* = 1$ ($u^* = 0, \pm 2\pi$, etc.), Eq. (A6) has one unique solution $v^* > 0$ fulfilling $v^* + \varepsilon \sinh v^* = \gamma$.

But for $\cos u^* = -1$, in contrast, the left-hand side of Eq. (A6) does not increase strictly with respect to v^* , so this equation can have multiple solutions. It turns out that there are three different positive solutions v^* for (approximately) subcritical $\gamma < \gamma_{\text{crit}} \approx |\ln \varepsilon|$, see Eq. (36), and one positive solution for supercritical γ . In the case of multiple singularities τ^* with the same real value u^* , the singularity which lies closest to the real axis dominates.

b. Movement and asymptotics of the singularities

In the subcritical regime, the regular singularity is well described by the Taylor series

$$\tau_{\text{reg}}^* = \gamma(i - p) - \sin[\gamma(i - p)]\varepsilon + \sin[2\gamma(i - p)]\frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3). \quad (\text{A10})$$

As in the Gauss pulse case, the singularity equation (34) can be solved asymptotically for $p = 0$ and $\varepsilon \rightarrow 0$ (thus $\text{Im } \tau_{\text{add}}^* \rightarrow \infty$) in order to describe the additional singularities. Here, we get

$$\tau_{\text{add},n}^* \approx \pi(2n - 1) + i|\ln \varepsilon| \quad (\text{A11})$$

with $n \in \mathbb{Z}$, i.e., each additional singularity converges against an edge of its “ π interval” for small ε ; see Fig. 5. Note that each n represents two different solutions of the singularity equation (\pm cases) again since these two solutions merge asymptotically.

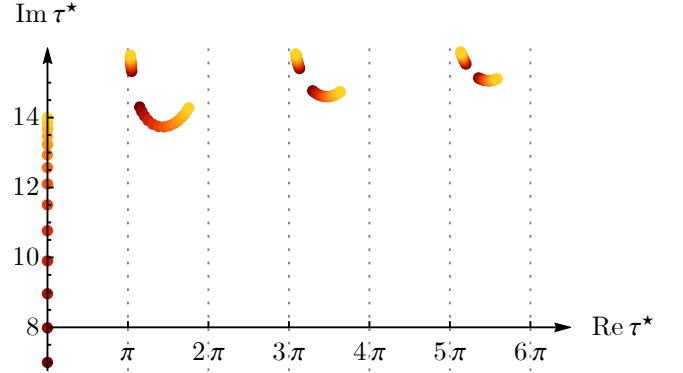
If we want to sum up the contributions from all singularities to the contour integral (13), we have to approximate the singularities for fixed parameters p , ε , and γ instead of fixed n . Concentrating on the mode $p = 0$ for simplicity and inspired by the graphical method in Fig. 4, we find that the real parts u^* of the additional singularities do asymptotically ($|u^*| \gg 1$) coincide with the centers of the “ π intervals”, i.e., $\tau_{\text{add}}^* \approx \tau_{\pm,n}^*$ for large $n \in \mathbb{N}$ with

$$\begin{aligned} \tau_{\pm,n}^* &= \pm\pi \left(2n - \frac{1}{2}\right) + i v_{\text{sing}} \left(\pm 2\pi n \mp \frac{\pi}{2}\right) \\ &= \pm\pi \left(2n - \frac{1}{2}\right) + i \text{arcosh} \left(\frac{2\pi n - \pi/2}{\varepsilon}\right). \end{aligned} \quad (\text{A12})$$

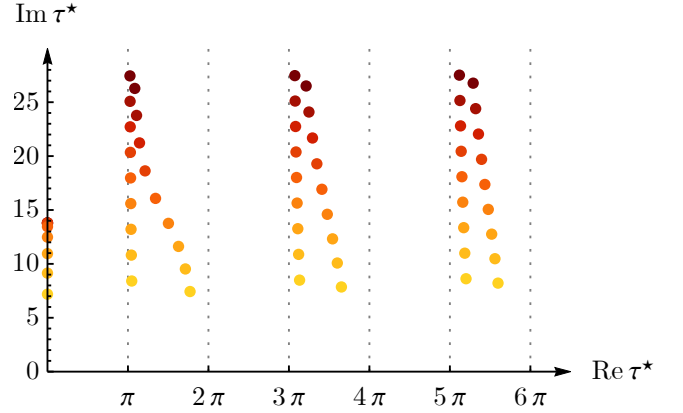
In this context, the \pm sign represents singularities with positive/negative real parts. Apart from that distinction, each n corresponds to two actual singularities again: the \pm cases of Eq. (A8), which merge in the considered limit. Since $\text{arcosh } x \approx \ln(2x)$ for $x \gg 1$, the imaginary part of the additional singularities grows logarithmically with n while their real part increases linearly.

Appendix B: Calculation of the contour integral (13)

In order to estimate the integral (13), the integration contour can be shifted away from the real axis, upwards into the complex plane, until the first singularity



(a) Singularities for constant $\varepsilon = 10^{-5}$ ($\gamma_{\text{crit}} \approx 11.5$) and γ growing from 7 to 20 (dark red to light yellow dots). The linear movement of the regular singularity $\tau_{\text{reg}}^* \approx i\gamma$ (ordinary Sauter-Schwinger effect) slows down exponentially near the critical threshold according to Eq. (35). The additional singularities move along the complex curves $u + i v_{\text{sing}}(u)$, see Fig. 4(b), which are independent of γ for $p = 0$.



(b) Moving singularities for $\gamma = 13.8$ and $\varepsilon = 10^s$ with $s = -10, \dots, -2$ (dark red to light yellow dots). The critical threshold is at $\varepsilon \approx 10^{-6}$. The regular singularity starts to approach the real axis as the nearby additional singularities come close. For decreasing ε , each additional singularity approaches the imaginary axis, converging towards the edge of its “ π interval”, cf. Eq. (A11).

FIG. 5: Numerical solutions τ^* of the singularity equation (34) for the oscillating weak field and $p = 0$.

is met at some t_k^* . Since the integrand is analytic for $0 < \text{Im } t < \text{Im } t_k^*$, this operation is permitted and the behavior of the complex phase $\varphi_k(t)$ gives an estimate of the integrand $\Xi_k(t) \exp[2i\varphi_k(t)]$ with now complex t . However, for the estimation of the integral, additional control of the integration along the line $\text{Im } t = \text{Im } t_k^*$ is required, which is rather difficult to obtain. Therefore, the integration contour is shifted further upwards, behind the singularities (see Fig. 6), and Cauchy’s theorem is applied.

The phase $\varphi_k(t)$ and the integrand can be analytically continued, however, locally on different Riemann sheets (with cuts in between) as the singularities are of a square-root type. The contribution from such a shifted integration contour is subdominant due to the exponen-

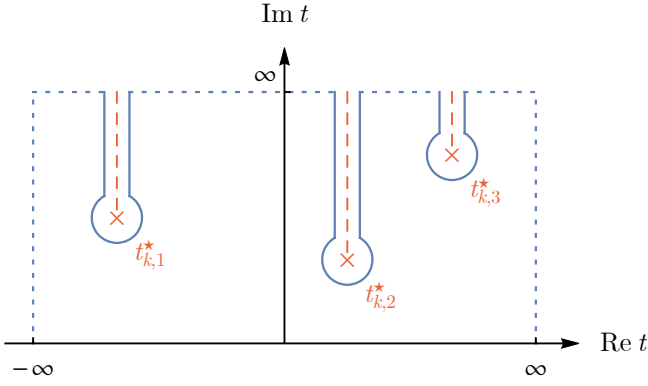


FIG. 6: Example for the deformed complex integration contour (dotted parts: no contribution; solid parts: branch cut contributions and residua) in the case of three singularities $t_{k,1-3}^*$ (crosses) with branch cuts running vertically upwards (dashed lines).

tial damping coming from the phase $\varphi_k(t)$ [Eq. (7)], but there remain contributions from integration along both sides of the cuts and around the singularities. In order to understand how they can be controlled, it is instructive to consider first the ordinary Sauter-Schwinger effect as an example.

c. Contour integral for a constant electric field E

The advantage of this example is that here contributions from the singularity and from the cut are explicitly determinable. Since the singularity t_k^* in $\Omega_k(t)$ —the plus case of Eq. (16) is relevant in this context—is of square-root type, the phase $\varphi_k(t)$, which is obtained as an integral of $\Omega_k(t)$, behaves as $(t - t_k^*)^{3/2}$. The residuum (representing a circle integral around the singularity) can be calculated exactly

$$S_p = -\frac{\pi}{2} \exp \left[-\left(\frac{\pi}{2} + i\phi(p) \right) \frac{E_S}{E} \right] \quad (\text{B1})$$

with the function $\phi(z)$ defined in Eq. (25). The complex cut emerging from that singularity, chosen vertically upwards from t_k^* , produces a jump in the phase function $\varphi_k(t)$, however, only in its real part. In consequence, the absolute value of the integrand $\Xi_k(t) \exp[2i\varphi_k(t)]$ is continuous across the cut and decays merely polynomially for $\text{Im } t > \text{Im } t_k^*$ due to $\Xi_k(t)$; it does not decay exponentially along the cut as might be expected from the behavior for $\text{Im } t < \text{Im } t_k^*$. However, the difference of values on both sides of the cut oscillates rapidly. The contribution from the cut can be written as

$$S_c = \exp \left[-\left(\frac{\pi}{2} + i\phi(p) \right) \frac{E_S}{E} \right] \int_0^\infty I_c(u) du \quad (\text{B2})$$

where $I_c(u) \approx \sin[2u^3 E_S/(3E)]/u$ is a fast-oscillating function of u . The integral converges solely due to its

fast oscillation and gives approximately the number $\pi/6$. In effect, both contributions to R_k^{out} ,

$$R_k^{\text{out}} = S_p + S_c \approx -\frac{\pi}{3} \exp \left[-\left(\frac{\pi}{2} + i\phi(p) \right) \frac{E_S}{E} \right], \quad (\text{B3})$$

from the singularity and along the cut are of the same order. We take it as suggestion that neglecting the contribution from the cuts in more complicated (time-dependent) electric fields should not change the final result by more than a constant numerical factor.

d. Contour integral for the oscillating weak field

In the case of the oscillating-weak-field profile given by Eqs. (32)–(33), there are infinitely many singularities as shown in Appendix A 2. Considering $p = 0$ again, we get two additional singularities per “ π interval” (see Fig. 4; the singularities with $\text{Re } \tau_{\text{add}}^* < 0$ are just mirror images of those with positive real parts) plus the regular singularity on the imaginary axis. In order to calculate the contour integral (13), we need to integrate around all these singularities (generating residua) and along both sides of the corresponding branch cuts. As the example above shows, it can be expected that the contributions from the cuts to R_k^{out} are of the same order as the residua $S_p(t_k^*)$ at the singularities $t_k^* = \tau^*/\omega_2$. Hence, R_k^{out} is approximately given by the residual sum

$$R_k^{\text{out}} \approx S_p(\tau_{\text{reg}}^*) + \sum_{\tau_{\text{add}}^*} S_p(\tau_{\text{add}}^*) \quad (\text{B4})$$

in dimensionless time units $\tau = \omega_2 t$. At each singularity, $\Xi_k(t)$ has a pole while $\varphi_k(t)$ has merely a square-root-type branch point, so $\varphi_k(t)$ may be treated as constant when integrating $\Xi_k(t) \exp[2i\varphi_k(t)]$ along a small circle around the pole. This way, we get

$$S_p(\tau^*) = \pm \frac{\pi}{2} e^{2i\varphi_k(\tau^*)} \quad (\text{B5})$$

where the sign depends on whether τ^* is a plus or minus solution of the singularity equation (34) and on the concrete potential $A(t)$. Although we cannot sum the series (B4) exactly, we can prove its convergence and estimate its value. Using the asymptotic form of the singularity positions τ_{add}^* for $|\text{Re } \tau_{\text{add}}^*| \gg 1$ and $p = k/m = 0$ [Eq. (A12)], we can approximate the values of φ_0 at most of the singularities and estimate the sum over all additional singularities appearing in Eq. (B4)

$$\left| \sum_{\tau_{\text{add}}^*} S_p(\tau_{\text{add}}^*) \right| \leq \sum_{\tau_{\text{add}}^*} |S_p(\tau_{\text{add}}^*)| = \frac{\pi}{2} \sum_{\tau_{\text{add}}^*} e^{-2 \text{Im}[\varphi_0(\tau_{\text{add}}^*)]}. \quad (\text{B6})$$

Taking into account the contributions from both series of singularities, i.e., the \pm cases in Eq. (34), we get

$$\sum_{\tau_{\text{add}}^*} e^{-2 \text{Im}[\varphi_0(\tau_{\text{add}}^*)]} \approx 2 \sum_{n=1}^{\infty} \left(\exp\{-2 \text{Im}[\varphi_0(\tau_{+,n}^*)]\} + \exp\{-2 \text{Im}[\varphi_0(\tau_{-,n}^*)]\} \right) \quad (\text{B7})$$

with $\tau_{\pm,n}^*$ from Eq. (A12). For singularities with large real parts $|\text{Re} \tau^*| \gg 1$, we neglect the m^2 term under the square root in $\Omega_k(t)$ [see Eq. (5)], and after some other minor analytic simplifications we get

$$\text{Im}[\varphi_0(\tau^*)] \approx \frac{E_S}{\gamma^2 E_1} |\text{Re} \tau^*| (\text{Im} \tau^* - 1). \quad (\text{B8})$$

Inserting it into the above sum and performing some further minor simplifications, we find the estimate

$$\sum_{\tau_{\text{add}}^*} e^{-2 \text{Im}[\varphi_0(\tau_{\text{add}}^*)]} < \frac{4 \left(\frac{3\pi E_1}{e E_2} \right)^{\pi q E_1 / \omega_2^2}}{\left(\frac{3\pi E_1}{e E_2} \right)^{4\pi q E_1 / \omega_2^2} - 1} \quad (\text{B9})$$

in terms of the usual physical quantities. The right-hand side is finite, which proves the convergence of the residual sum (at least for $p = 0$). As a check, two limiting cases can be considered:

- a) $E_2 \rightarrow 0$: As the assisting oscillation vanishes, we expect the ordinary Sauter-Schwinger effect. Consistently, the right-hand side of Eq. (B9) becomes zero in this limit—there is no contribution from the additional singularities.
- b) $\omega_2^2 \rightarrow 0$: If the oscillation becomes a static field which has a negligible influence due to $E_2 \ll E_1$, the right-hand side of Eq. (B9) vanishes again, leaving the ordinary Sauter-Schwinger effect, as expected.

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- [1] F. Sauter, Z. Phys. **69**, 742 (1931); **73**, 547 (1932).
 - [2] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
 - [3] V. Weisskopf, Kong. Dans. Vid. Selsk., Math.-fys. Medd. **XIV**, 6 (1936), reprinted in Quantum Electrodynamics, edited by J. Schwinger (Dover, New York, 1958).
 - [4] J. Schwinger, Phys. Rev. **82**, 664 (1951).
 - [5] In contrast to electron-positron pair creation in the perturbative multiphoton regime; see Ref. [6].
 - [6] D. L. Burke *et al.*, Phys. Rev. Lett. **79**, 1626 (1997).
 - [7] R. Schützhold, H. Gies, and G. Dunne, Phys. Rev. Lett. **101**, 130404 (2008).
 - [8] M. Ruf, G. R. Mocken, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. **102**, 080402 (2009).
 - [9] F. Hebenstreit, R. Alkofer, G. V. Dunne and H. Gies, Phys. Rev. Lett. **102**, 150404 (2009).
 - [10] G. V. Dunne, H. Gies, and R. Schützhold, Phys. Rev. D **80**, 111301 (2009).
 - [11] A. Monin and M. B. Voloshin, Phys. Rev. D **81**, 025001 (2010).
 - [12] G. R. Mocken, M. Ruf, C. Müller, and C. H. Keitel, Phys. Rev. A **81**, 022122 (2010).
 - [13] A. Monin and M. B. Voloshin, Phys. Rev. D **81**, 085014 (2010).
 - [14] C. K. Dumlu, Phys. Rev. D **82**, 045007 (2010).
 - [15] C. K. Dumlu and G. V. Dunne, Phys. Rev. D **83**, 065028 (2011).
 - [16] M. Orthaber, F. Hebenstreit and R. Alkofer, Phys. Lett. B **698**, 80 (2011).
 - [17] C. Fey and R. Schützhold, Phys. Rev. D **85**, 025004 (2012).
 - [18] M. Jiang, W. Su, Z. Q. Lv, X. Lu, Y. J. Li, R. Grobe, and Q. Su, Phys. Rev. A **85**, 033408 (2012).
 - [19] A. Nuriman, B.-S. Xie, Z.-L. Li, and D. Sayipjamal, Phys. Lett. B **717**, 465 (2012); Commun. Theor. Phys. **59**, 331 (2013).
 - [20] C. Kohlfürst, M. Mitter, G. von Winckel, F. Hebenstreit, and R. Alkofer, Phys. Rev. D **88**, 045028 (2013).
 - [21] A. Nuriman, Z.-L. Li, and B.-S. Xie, Phys. Lett. B **726**, 820 (2013).
 - [22] M. J. A. Jansen and C. Müller, Phys. Rev. A **88**, 052125 (2013).
 - [23] C. K. Dumlu, Phys. Rev. D **89**, 065011 (2014).
 - [24] A. Blinne and H. Gies, Phys. Rev. D **89**, 085001 (2014).
 - [25] Z. L. Li, D. Lu, B. S. Xie, L. B. Fu, J. Liu, and B. F. Shen, Phys. Rev. D **89**, 093011 (2014).
 - [26] E. Strobel and S.-S. Xue, Nucl. Phys. **B886**, 1153 (2014).
 - [27] I. Akal, S. Villalba-Chávez, and C. Müller, Phys. Rev. D **90**, 113004 (2014).
 - [28] F. Hebenstreit and F. Fillion-Gourdeau, Phys. Lett. B **739**, 189 (2014).
 - [29] A. Otto, D. Seipt, D. Blaschke, B. Kämpfer, and S. A. Smolyansky, Phys. Lett. B **740**, 335 (2015); A. Otto, D. Seipt, D. Blaschke, S. A. Smolyansky, and B. Kämpfer, Phys. Rev. D **91**, 105018 (2015).
 - [30] C. Schneider and R. Schützhold, [arXiv:1407.3584](#).
 - [31] There are also studies of scenarios where pair creation is assisted by nuclear fields (see, e.g., Ref. [32]) but we do not consider this case here.
 - [32] A. Di Piazza, E. Lötstedt, A. I. Milstein, and C. H. Keitel, Phys. Rev. Lett. **103**, 170403 (2009); S. Augustin and C. Müller, Phys. Lett. B **737**, 114 (2014).
 - [33] L. C. Baird, J. Math. Phys. **11**, 2235 (1970); J. P. Davis and P. Pechukas, J. Chem. Phys. **64**, 3129 (1976); see also S. Massar and R. Parentani, Nucl. Phys. **B513**, 375 (1998) and references therein.

- [34] C. K. Dumlu and G. V. Dunne, Phys. Rev. D **84**, 125023 (2011).
- [35] E. Akkermans and G. V. Dunne, Phys. Rev. Lett. **108**, 030401 (2012).
- [36] L. V. Keldysh, Sov. Phys. JETP **20**, 1307 (1965).
- [37] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 2007).
- [38] R. P. Feynman, Phys. Rev. **80**, 440 (1950).
- [39] I. K. Affleck, O. Alvarez, and N. S. Manton, Nucl. Phys. **B197**, 509 (1982).
- [40] S. P. Kim and D. N. Page, Phys. Rev. D **65**, 105002 (2002).
- [41] G. V. Dunne and C. Schubert, Phys. Rev. D **72**, 105004 (2005).
- [42] G. V. Dunne, Q. H. Wang, H. Gies, and C. Schubert, Phys. Rev. D **73**, 065028 (2006).
- [43] S. P. Kim, D. N. Page, Phys. Rev. D **73**, 065020 (2006).
- [44] G. V. Dunne, J. Phys. A **41**, 164041 (2008).
- [45] C. Schubert, Lectures on the worldline formalism, in *Proceedings of the School on Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins and Conformal Geometry, Morelia, Mexico 2012* (to be published), <http://indico.cern.ch/event/206621/>.
- [46] A. Ilderton, G. Torggrimsson, and J. Wårdh, Phys. Rev. D **92**, 065001 (2015).
- [47] Note that one should carefully distinguish between the perturbative effect of vacuum polarization during the pulse—which can be interpreted as a suitably defined intermediate particle number—and the nonperturbative Sauter-Schwinger effect corresponding to the number of created particles after the pulse; see also Ref. [48].
- [48] J. Zahn, [arXiv:1501.06527](https://arxiv.org/abs/1501.06527).
- [49] C. Schneider and R. Schützhold (to be published).